

Dispersion relations with crossing symmetry for $\pi\pi$ D and F wave amplitudes

R. Kamiński

*Department of Theoretical Physics Henryk Niewodniczański Institute of Nuclear Physics,
Polish Academy of Sciences, 31-342, Kraków, Poland*

A set of once subtracted dispersion relations with imposed crossing symmetry condition for the $\pi\pi$ D - and F -wave amplitudes is derived and analyzed. An example of numerical calculations in the effective two pion mass range from the threshold to 1.1 GeV is presented. It is shown that these new dispersion relations impose quite strong constraints on the analyzed $\pi\pi$ interactions and are very useful tools to test the $\pi\pi$ amplitudes. One of the goals of this work is to provide a complete set of equations required for easy use. Full analytical expressions are presented. Along with the well known dispersion relations successful in testing the $\pi\pi$ S - and P -wave amplitudes, those presented here for the D and F waves give a complete set of tools for analyzes of the $\pi\pi$ interactions.

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I. INTRODUCTION

The introduction of crossing symmetry conditions into dispersion relations for the $\pi\pi$ amplitudes was promoted by Roy in 1971 [1]. Since then, now well-known the Roy's equations have been successfully used to test the $\pi\pi$ S - and P -wave amplitudes [2–12]. These equations with two subtractions have been used for example to analyze the low energy $\pi\pi$ interaction parameters [5] and to eliminate the long standing "up-down" ambiguity in scalar-isoscalar $\pi\pi$ wave amplitudes [3, 6]. Quite recently interest in these equations has increased significantly due to series of works by Bern and Madrid groups [7–10]. In these analyzes, authors have used, inter alia, the Roy's equations to construct the $\pi\pi$ S and P wave amplitudes fulfilling crossing symmetry for effective two pion mass $m_{\pi\pi}$ from the threshold to over 1 GeV. To perform it they had to describe the $\pi\pi$ S -, P -, D -, F - and G -wave amplitudes using phenomenological parameterizations below 1.42 GeV [10] or 2 GeV [8] and Regge amplitudes up to several GeV. In such a way these analyzes delivered prescription for constructions of unitary $\pi\pi$ amplitudes for many partial waves in very wide energy ranges. Together with the outcome of new precise data near the $\pi\pi$ threshold [13] these works led also to very accurate determination of the mass and width of the $f_0(600)$ (or σ) resonance and of the threshold parameters [4, 11].

Very recently the $\pi\pi$ S and P wave amplitudes with crossing symmetry constraints have been analyzed using not only the Roy's equations but also once subtracted dispersion relations [11, 12]. Due to one less subtraction these new equations, called GKPY, proved to be much more demanding than the Roy ones. Above around $m_{\pi\pi} = 450$ MeV the GKPY equations have much smaller uncertainties so they impose stronger constraints on the $\pi\pi$ amplitudes.

In the works mentioned above only the S and P wave amplitudes have been directly fitted to the Roy's or GKPY equations. The higher partial wave amplitudes have been fitted only indirectly via their relations with the S and P waves within the dispersive equations.

In this paper are derived once subtracted dispersion relations with imposed crossing symmetry condition for D and F waves, hereafter called $OSDR_{DF}$. Together with the Roy's and GKPY equations they will form a complementary set of dispersion relations which can be very useful in testing the $\pi\pi$ S -, P -, D - and F -wave amplitudes.

This paper is organized as follows: in Section II general structure and analytical properties of the $OSDR_{DF}$ are presented and discussed. Individual components of these equations are analyzed in detail. The third section contains an example of numerical calculations. Full derivation of $OSDR_{DF}$ and analytical expressions for their components are given in Appendices A and B. Discussion of results and summary are in Section IV.

Through the text, partial waves with orbital momentum l and isospin I are denoted by WI or just by W if the isospin does not need to be specified. The W can be S , P , D , F or G for l equal to 0, 1, 2, 3 or 4, respectively.

II. DISPERSION RELATIONS

Dispersion relations, for example the Roy's and GKPY ones, relate real parts of given partial wave amplitudes with sets of imaginary parts of other ones. General form of dispersion relations with one subtraction for the D - and F -wave amplitudes reads:

$$\begin{aligned} \text{Re } f_\ell^I(s) = & -\frac{1}{24}(a_0^0 - \frac{5}{2}a_0^2)\delta_{I1}\delta_{I3} + \\ & \sum_{I'=0}^2 \sum_{\ell'=0}^3 \int_{4m_\pi^2}^{s'_{max}} ds' K_{\ell\ell'}^{II'}(s, s') \text{Im } f_{\ell'}^{I'}(s') + d_\ell^I(s) \end{aligned} \quad (1)$$

where $s = m_{\pi\pi}^2$. The first component $-(a_0^0 - 2.5a_0^2)/24$ is so called subtracting term ST_ℓ^I . Two remaining parts are called the kernel $KT_\ell^I(s)$ and driving $DT_\ell^I(s)$ terms.

Below $s' = s'_{max}$ partial wave amplitudes $f_\ell^I(s)$ can be expressed by their relations with the experimental $\pi\pi$

phase shifts $\delta_l^I(s)$ and inelasticities $\eta_l^I(s)$

$$f_l^I(s) = \frac{\eta_l^I(s)e^{i\delta_l^I(s)} - 1}{2i\sigma(s)} \quad (2)$$

where

$$\sigma(s) = \sqrt{\frac{s - 4m_\pi^2}{s}}. \quad (3)$$

For brevity hereafter the real parts of amplitudes on the left side of Eq. (1) will be called "output amplitudes" and the imaginary parts of amplitudes on the right side - "input amplitudes".

The angular momentum l' of the input amplitudes goes from 0 to 3 (the S , P , D and F waves). As was shown in [12] the input amplitude for the G wave ($l' = 4$) is very small and has negligible influence on the output. Because of the Bose symmetry the sums $l' + I'$ and $l + I$ for input and output amplitudes respectively, must be even.

As presented in Appendix A the subtracting terms ST_ℓ^I in once subtracted dispersion relations are constant and are determined by combinations of real parts of partial wave amplitudes at the $\pi\pi$ threshold. As can be seen from threshold expansion

$$Re f_l^I(k) = k^{2l} \left(a_l^I + b_l^I k^2 + O(k^4) \right) \quad (4)$$

where the pion momentum $k = \sqrt{(s/4 - m_\pi^2)}$, the only nonzero amplitudes at $k = 0$ are those for the S_0 and S_2 waves. Their values at the threshold are scattering lengths a_0^0 and a_0^2 , respectively. In Appendix A it is shown that in the case of dispersion relations for the D and F partial waves the only nonzero combination of these two scattering lengths in the subtracting terms is for the F wave. It is due to nonzero integral (from 0 to 1) of the Legendre polynomial for this partial wave. The scattering lengths a_0^0 and a_0^2 can be treated as an input and may be fixed using for example ChPT predictions (see e.g. [8]) or can be fitted to data and to theoretical constraints (see e.g. [10–12]).

In the kernel terms $KT_\ell^I(s)$ the products of the input amplitudes and of the kernels $K_{ll'}^{II'}(s, s')$, defined in Appendix A, are integrated over s' from the $\pi\pi$ threshold to s'_{max} i.e. up to the energy where the phenomenological parameterizations of the phase shifts and inelasticities in input amplitudes (see Eq. (2)) are quite well known. In practice $s'_{max} \sim 1.4 - 2$ GeV (see e.g. analyzes of the Roy's [8] and GKPY equations [12]).

Above $s' = s'_{max}$ the input amplitudes are parameterized using the Regge formalism. Integrals in this s' range are grouped into the driving terms $DT_\ell^I(s)$. Contrary to the kernel terms, the products of the input amplitudes and corresponding kernels in $DT_\ell^I(s)$ must be doubly integrated - over s' and t . This is due to the t dependence of the Regge amplitudes.

As shown in Appendix A, the only singularities in Eq. (1) are those at $s' = s$ in the diagonal kernel elements

i.e for $l = l'$. Therefore one has to take principal value there.

As was already pointed out in [12] while comparing the GKPY and Roy's equations for the S and P waves, an essential feature of the once subtracted dispersion relations is their slower convergence than in twice subtracted dispersion relations. In the case of one subtraction, the integrands in the $KT_\ell^I(s)$ and in $DT_\ell^I(s)$ behave as $1/s'^2$ for $s' \rightarrow \infty$ while in the Roy's equations as $1/s'^3$. Due to this difference the input amplitudes at higher energies in the former equations enter with higher weights than in the latter ones. It is especially important for higher partial waves which contribute mainly above 1 GeV and therefore are undervalued in relations with two subtractions. Since the output amplitudes for given l partial wave depend also on the input ones with $l' = l$ this argument is reinforced in this analysis dealing with D and F output partial wave amplitudes.

Constant value of the ST_ℓ^I in $OSDR_{DF}$ leads to constant value of its errors which, one can expect, are much smaller (for higher s) than these which would be in analogous relations with two subtractions. It is due to the fact that subtracting terms in these relations are not constant but are linear functions of s which leads to increase of their uncertainties with energy. Detailed comparison of the errors in the once and twice subtracted dispersion relations for the S and P waves (the GKPY and Roy's equations, respectively) can be found in [12].

The application range of the $OSDR_{DF}$, Eq. (1), is the same as of the GKPY equations in [12] and comes about 1.1 GeV. One can expect that below this energy only the D_0 output amplitude will show a clear increase with the energy due to presence of the $f_2(1270)$ resonance. The absence of any resonance in the D_2 wave and the relatively large mass of meson $\rho_3(1690)$ in the F_1 one, lead to rather small variation of amplitudes in these waves below 1.1 GeV.

Wave	TE	ST_ℓ^I	$KT_\ell^I(s)$	$DT_\ell^I(s)$
D_0	0	0	$\alpha_0 + c_0\beta_0$	$\gamma_0 + d_0\delta_0$
F_1	0	$-\frac{1}{24}(a_0^0 - \frac{5}{2}a_0^2)$	$A + \alpha_1 + \beta_1$	$B + \gamma_1 + \delta_1$
D_2	0	0	$\alpha_2 + c_2\beta_2$	$\gamma_2 + d_2\delta_2$

TABLE I: Comparison of the threshold expansion TE (Eq. (4)) results for $s \rightarrow 4m_\pi^2$ and of the threshold behavior of subtracting ST_ℓ^I , kernel $KT_\ell^I(s)$ and driving $DT_\ell^I(s)$ terms for D and F partial waves. The α_I and γ_I are terms of the order $O(s - 4m_\pi^2)$ and β_I , δ_I of the order $O(s - 4m_\pi^2)^2$. Values of the A , B , c_I and d_I constants are explained in the text.

In practical applications of the $OSDR_{DF}$ it is very interesting and useful to compare the threshold behavior of the ST_ℓ^I , $KT_\ell^I(s)$ and $DT_\ell^I(s)$ for D and F waves. According to the threshold expansion (4), the sum of these components should vanish at $s = 4m_\pi^2$ for all these waves. Comparison of the threshold expansions of the subtracting, kernel and driving terms is presented in Table I. For the D_0 and D_2 waves the zero order parts in all components are equal to zero, which automatically ensures cor-

rect behavior of the full output amplitudes at $s = 4m_\pi^2$. Of course sum of the first order parts α_I and γ_I must be equal to zero and sum of the c_I and d_I should give corresponding scattering lengths. In case of the $F1$ wave the nonzero value of subtracting term must be canceled by sum of nonzero values of $KT_\ell^I(4m_\pi^2)$ and $DT_\ell^I(4m_\pi^2)$. Sums of the first and second order parts should give zero.

It is worthy to emphasize here that such cancellations

demand strong and proper mutual relations between the amplitudes for all waves integrated over very wide energy range in the $KT_\ell^I(s)$ and $DT_\ell^I(s)$. These relations for the zero order parts of the $F1$ wave, i.e. for A and B in Table I, involve also the threshold parameters of the lowest partial waves ($S0$ and $S2$) and are expressed by the well known Olsson sum rule [14]

$$\begin{aligned} (a_0^0 - \frac{5}{2}a_0^2) = & 2 \int_{4m_\pi^2}^{s'_{max}} \frac{ds'}{\pi s'(s' - 4m_\pi^2)} \left[2Imf_0^0(s') + 10Imf_2^0(s') + 9Imf_1^1(s') + 21Imf_3^1(s') - 5Imf_0^2(s') - 25Imf_2^2(s') \right] \\ & + 12 \int_{s'_{max}}^{\infty} \frac{ImF_t^1(s', 0)}{\pi s'(s' - 4m_\pi^2)} ds' \end{aligned} \quad (5)$$

where $F_t^1(s', t)$ is the isospin 1 amplitude in the t channel (see Appendix A).

Of course an identical relation may also be obtained from the GKPY equations. In practice this sum rule gives a chance to verify an accuracy of used parameterizations of many input partial wave amplitudes in a very wide energy range and validates the choice (or result of a fit) of scattering lengths a_0^0 and a_0^2 .

III. NUMERICAL ANALYSIS

As an example of practical application of the $OSDR_{DF}$, the D and F output amplitudes have been calculated using the input ones from [12]. These input amplitudes were obtained there in dispersive data analysis using forward dispersion relations, sum rules, the Roy's and GKPY equations. The D and F waves have not been directly fitted to any dispersion relation.

Figures 1 to 3 present $m_{\pi\pi}$ distributions of the output amplitudes and the real parts of the amplitudes whose imaginary parts have been used as the inputs in Eq. (1). The error bands of the output amplitudes have been calculated using Monte Carlo method for all 53 parameters used in [12] to parameterize the low and high energy behavior of all input amplitudes. Assuming Gaussian distributions of these parameters they have been varied randomly 10^3 times within their left and right 3σ which resulted in three Gaussian distributions for the three output amplitudes. This procedure has been carried out for 25 values of s between the $\pi\pi$ threshold and 1110 MeV. The errors of the output amplitudes have been determined by widths of the Gaussian functions fitted to the left and right sides of these distributions at each s independently. Of course, due to the large number of parameters, these errors are very similar to those obtained as square root of sum of squares of individual errors of

each parameter. However, the Monte Carlo method has been chosen to present possible asymmetries of final errors caused by correlations between varied parameters.

As is seen on the figures 1 to 3 the error bands of the output $D0$ and $D2$ wave amplitudes go to zero for $s \rightarrow 4m_\pi^2$. The nonzero errors of the $F1$ wave at the threshold are due to the nonzero value of subtracting term and in fact represent its error. In Table I this constant ST_ℓ^I is a linear combination of two scattering lengths of the $S0$ and $S2$ waves. Therefore, the full error of the output $F1$ wave amplitude at the threshold is completely determined by the uncertainties of these two threshold parameters. Convergence of the $D0$ and $D2$ output amplitudes to 0 for $s \rightarrow 4m_\pi^2$ is well seen. In the case of the $F1$ wave, the output amplitude goes to $(-1 \pm 7) \cdot 10^{-4}$ which is compatible with the predicted zero value in Table I. The error has been calculated in the same way as the errors of the output amplitudes.

As the D and F wave amplitudes have not been directly fitted to any dispersion relation in [12], the fact that their input and output amplitudes for $m_{\pi\pi} > 800$ MeV differ by more than one σ is not surprising. One can expect, however, that the use of the presented here dispersion relations in more complete analysis of the $\pi\pi$ amplitudes e.g. in the analysis of the Bern or Madrid group would decrease this difference significantly. It would also increase the weight of the theoretical constraints imposed on these and on other wave amplitudes, and therefore would diminish their uncertainties.

Vastly different scales in the figures for the $D0$ and for $D2$ and $F1$ waves reflect significant differences in the sizes of the amplitudes. As was explained in Section II these differences are due to the presence of the meson $f_2(1270)$ in the $D0$ wave close to the studied s region and lack of such states in the $D2$ and $F1$ waves.

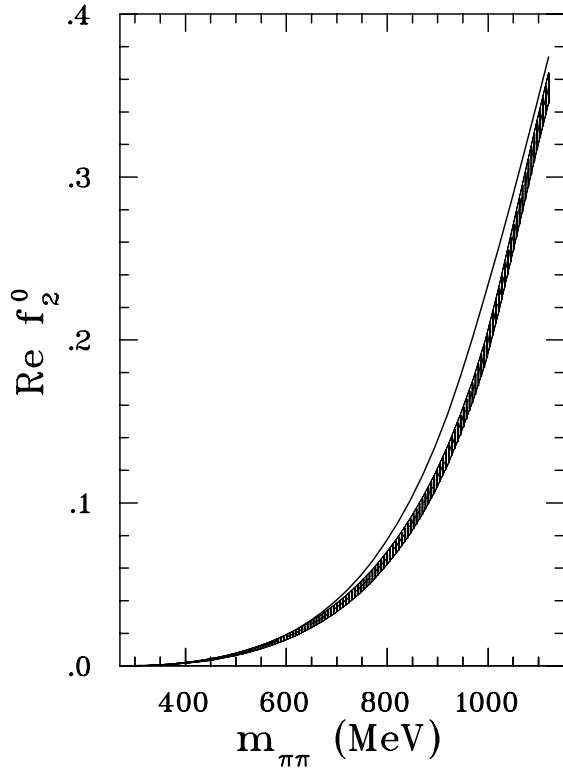


FIG. 1: Input (solid line) and output (dashed line) for the $D0$ wave amplitude in Eq. (1). Gray band represents errors of the output.

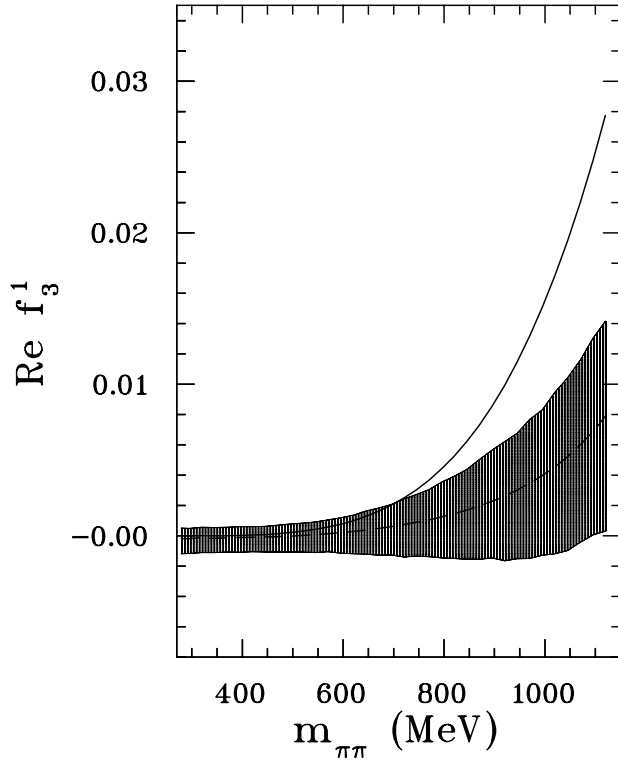


FIG. 2: As in Fig. 1 but for $F1$ wave amplitude.

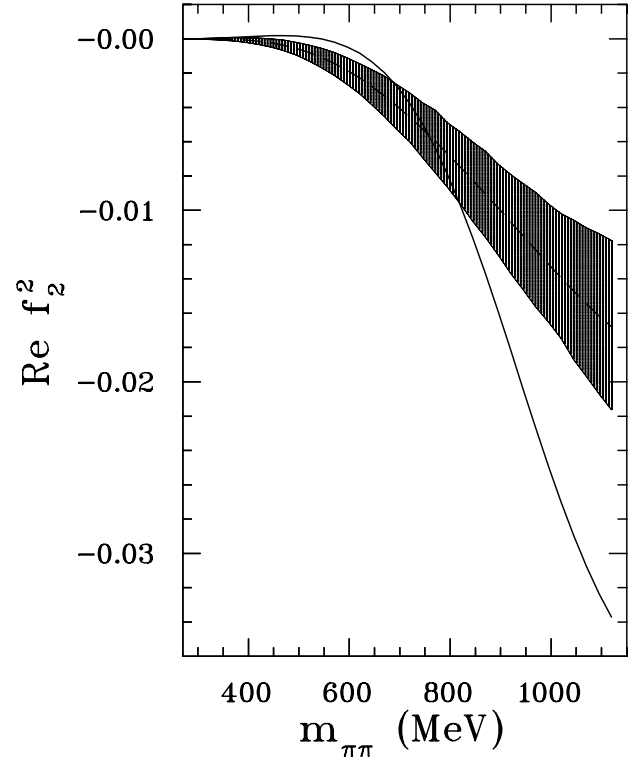


FIG. 3: As in Fig. 1 but for $D2$ wave amplitude.

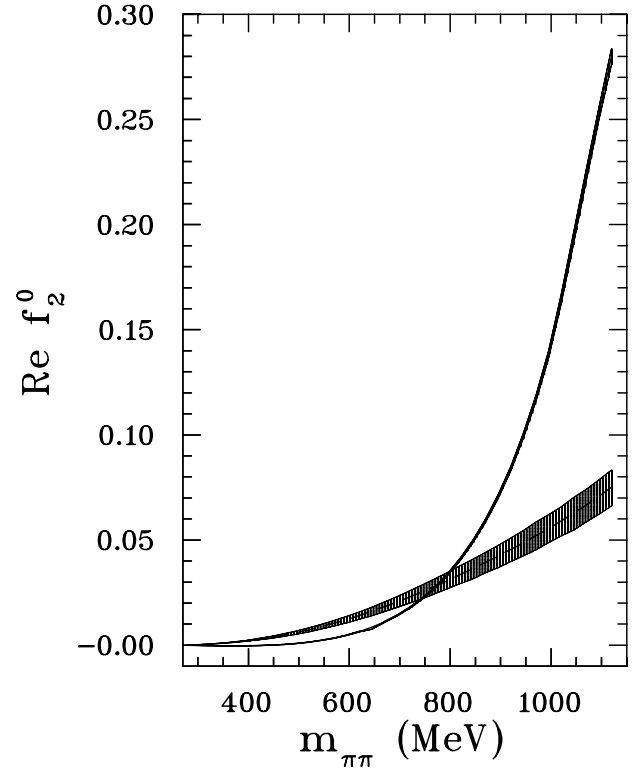


FIG. 4: Components of the output $D0$ wave amplitude: kernel term (solid line) and driving term (dashed line). Gray bands represent their errors.

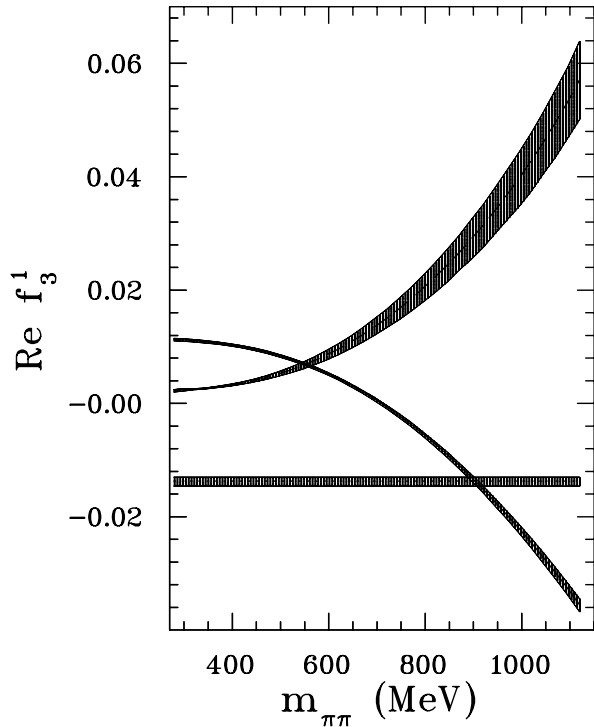


FIG. 5: As in Fig. 4 but for the $F1$ amplitude. Horizontal dashed-dotted line represents subtracting term.

Figures. 4 to 6 present the $m_{\pi\pi}$ dependences of the ST_ℓ^I , $KT_\ell^I(s)$ and $DT_\ell^I(s)$ components for the D and F wave amplitudes. Error bands have been calculated for each term separately as for the output amplitudes in Figs. 1 to 3. Again, due to the presence of the $f_2(1270)$ resonance much larger values of the $KT_\ell^I(s)$ and $DT_\ell^I(s)$ for the $D0$ wave than for the $D2$ and $F1$ ones are visible (note different scales on the figures).

For all three waves the errors of the kernel terms are significantly smaller than those of the driving ones. This is due to the fact that the input amplitudes in $KT_\ell^I(s)$ are more precisely known than those in $DT_\ell^I(s)$. The errors of the latter are determined by errors of the input amplitudes above s'_{max} . These amplitudes are, however, less precisely known than those at lower energies. In particular, small errors of the kernel for the $D0$ wave amplitude are due to presence of the well known resonance $f_2(1270)$.

In Fig. 5 for the $F1$ wave, the straight line represents nonzero subtracting term value of $-0.0137^{+0.0007}_{-0.0009}$. Its errors are completely determined by the uncertainties of the scattering lengths of the $S0$ and $S2$ waves. If the amplitudes of these waves at the threshold are not fixed but fitted (as it is e.g. in [12]) then of course, these uncertainties become functions of the errors of amplitude parameters. According to what was shown in Table I, the kernel and driving terms also have nonzero values at the threshold to compensate the subtracting term. Threshold values of the kernel and driving terms are: $A = 0.0113 \pm 0.0002$ and $B = 0.0023 \pm 0.0002$, respectively.

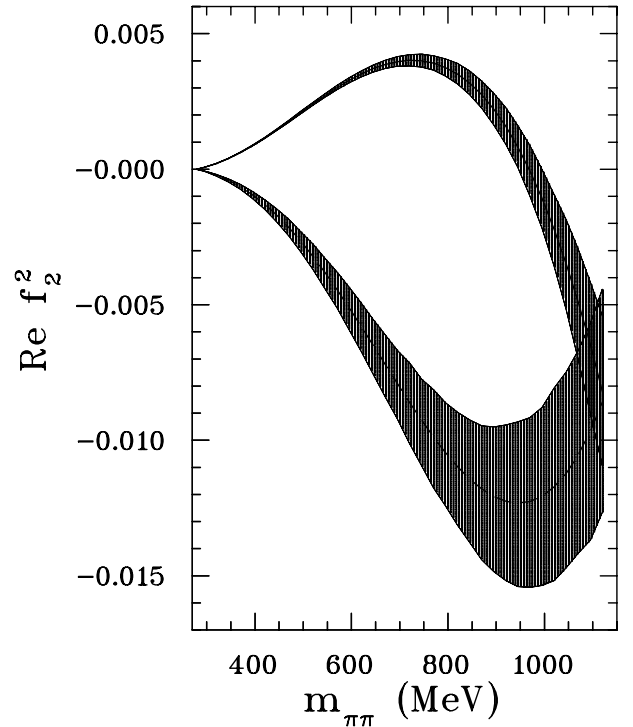


FIG. 6: As in Fig. 4 but for the $D2$ wave amplitude.

IV. CONCLUSIONS

A set of once subtracted dispersion relations with imposed crossing symmetry condition for the $\pi\pi$ D - and F -wave amplitudes has been derived and analyzed. Analytical structure of these equations and of their components have been studied and described in detail. It was shown that integrals in these equations converge slower than integrals in the twice subtracted dispersion relations (e.g. in the Roy's ones for the S and P partial waves). Thanks to this, the input amplitudes are less suppressed at higher energies which is essential for higher partial waves becoming significant only above 1 GeV. One can expect that use of the one subtraction will lead to the generation of smaller uncertainties of the output amplitudes than those created using two subtractions. In the latter case, the errors of the subtracting terms, being not constant but linear functions of s , grow with s . Therefore, the one subtracted dispersion relations provide more demanding tests for the $\pi\pi$ amplitudes.

Apart of the derivation of these equations, a first practical application has been presented. The input amplitudes in the low and high energy region have been taken from fit to experimental data and from direct, for the S and P wave amplitudes, and indirect fit to dispersive theoretical constraints in [12]. It has been shown that because of the D - and F -wave amplitudes were not fitted directly to any theoretical constraints there, their inputs and outputs, calculated in this paper, differ sizable above about 800 MeV. One can expect, however, that the com-

patibility between them will be significantly improved if presented here dispersion relations are used in fits of the $\pi\pi$ amplitudes.

It was shown that nonzero value of the subtracting term in the F^I output amplitude creates opportunity to relate low and high energy behavior of many partial wave amplitudes with the threshold parameters of the lowest ones. Therefore one can expect that these equations may help to decrease uncertainties of the $\pi\pi$ amplitudes and errors of these threshold parameters. One can also expect that the theoretical constraints imposed by these equations directly on the D and F wave amplitudes and indirectly on lower ones can lead to slightly more precise determination of the $f_0(600)$ and $f_0(980)$ parameters.

In conclusion the above derived and analyzed set of equations is easy to use and can be very helpful in future analyzes of the $\pi\pi$ interactions. Together with dispersion relations for the S and P waves - the Roy's and GKPY equations with two and one subtraction, respectively, they can be used as complementary set of equations to test $\pi\pi$ amplitudes from the threshold to about 1.1 GeV.

V. ACKNOWLEDGMENTS

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Appendix A: Derivation of the once subtracted dispersion relations

Let us define following isovector for the scattering amplitudes $F^I(s, t)$ of isospin I

$$\vec{F}(s, t) = \begin{pmatrix} F^0(s, t) \\ F^1(s, t) \\ F^2(s, t) \end{pmatrix}. \quad (\text{A1})$$

Then a once subtracted dispersion relations can be expressed by

$$\begin{aligned} \text{Re } \vec{F}(s, t) &= \text{Re } \vec{F}(s_0, t) + \frac{s - s_0}{\pi} \\ &\times \left[\int_{4m_\pi^2}^{\infty} ds' \frac{\text{Im } \vec{F}(s', t)}{(s' - s_0)(s' - s)} + \int_{-t}^{-\infty} ds' \frac{\text{Im } \vec{F}(s', t)}{(s' - s_0)(s' - s)} \right] \end{aligned} \quad (\text{A2})$$

where s_0 is the subtraction point to be defined later.

The first and second integral in Eq. (A2) are taken on the real s axis along the right and left hand cuts of $F^I(s, t)$, respectively. For any s values along these cuts, one should take the principal values for these integrals (see e.g. the discussion in Appendix B of [12]).

Performing the substitution

$$u' = 4m_\pi^2 - s' - t \quad (\text{A3})$$

and using the crossing symmetry relation

$$\vec{F}(u', t) = \hat{C}_{su} \vec{F}(s', t), \quad (\text{A4})$$

the left hand cut integrals can be recast in terms of the right hand cut ones

$$\begin{aligned} \text{Re } \vec{F}(s, t) &= \text{Re } \vec{F}(s_0, t) + \frac{s - s_0}{\pi} \\ &\times \int_{4m_\pi^2}^{\infty} ds' \left[\frac{\text{Im } \vec{F}(s', t)}{(s' - s_0)(s' - s)} - \hat{C}_{su} \frac{\text{Im } \vec{F}(s', t)}{(s' - u_0)(s' - u)} \right] \end{aligned} \quad (\text{A5})$$

with $u = 4m_\pi^2 - s - t$, $u_0 = 4m_\pi^2 - s_0 - t$ and with the crossing symmetry matrix \hat{C}_{su} defined in Eq. (A8).

Using the $s \longleftrightarrow t$ crossing symmetry relation one can express the subtracting terms $F^I(s_0, t)$ by

$$\vec{F}(s_0, t) = \hat{C}_{st} \vec{F}(t, s_0). \quad (\text{A6})$$

Following now the same procedure which was used in derivation of the $\text{Re } F^I(s, t)$ in Eq. (A5), one can get similar, once subtracted dispersion relations for $F^I(t, s_0)$:

$$\begin{aligned} \text{Re } \vec{F}(t, s_0) &= \text{Re } \vec{F}(t_0, s_0) + \frac{t - t_0}{\pi} \\ &\times \int_{4m_\pi^2}^{\infty} ds' \left[\frac{\text{Im } \vec{F}(s', s_0)}{(s' - t_0)(s' - t)} - \hat{C}_{su} \frac{\text{Im } \vec{F}(s', s_0)}{(s' - u_0)(s' - u)} \right] \end{aligned} \quad (\text{A7})$$

with $u_{00} = 4m_\pi^2 - s_0 - t_0$ and $t_0 = 4m_\pi^2 - s_0 - u$. The crossing matrices \hat{C}_{su} and \hat{C}_{st} used in Eqs. (A4) to (A7) read

$$\hat{C}_{su} = \begin{pmatrix} \frac{1}{3} & -1 & \frac{5}{3} \\ -\frac{1}{3} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}, \quad \hat{C}_{st} = \begin{pmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix}. \quad (\text{A8})$$

It is worthy to mention here that the set of equations (A7) can be easily obtained from (A5) using the following replacements: $s \rightarrow t$, $t \rightarrow s_0$, $s_0 \rightarrow t_0$, $u \rightarrow u_0$ and $u_0 \rightarrow u_{00}$.

The problem of the convergence of integrals (A5) and (A7) has been analyzed and described in [12]. It was shown there that, due to the Pomeron contribution from the $I_t = 0$ channel, the integrals along the left and right hand cuts are divergent when taken separately. However, taking both cuts into account simultaneously, a cancellation occurs and the integrands decay as $1/s'^2$ when $s' \rightarrow \infty$ which ensures convergence of the integrals. In the Roy's equations with two subtractions corresponding integrands behave like $1/s'^3$ which ensures convergence of each integral separately and of course faster, than in case of once subtracted equations, convergence of their sum.

Substituting now (A6) and (A7) into (A5) and following the Roy's original choice: $t_0 = 4m_\pi^2$ and $s_0 = 0$ one gets

$$\begin{aligned}
\text{Re } \vec{F}(s, t) = & \omega \hat{C}_{st} \vec{a} \\
& + \frac{t - 4m_\pi^2}{\pi} \int_{4m_\pi^2}^{\infty} ds' \hat{C}_{st} \left[\frac{\text{Im } \vec{F}(s', 0)}{(s' - t)(s' - 4m_\pi^2)} - \frac{\hat{C}_{su} \text{Im } \vec{F}(s', 0)}{s'(s' - u_0)} \right] \\
& + \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \left[\frac{\text{Im } \vec{F}(s', t)}{s'(s' - s)} - \frac{\hat{C}_{su} \text{Im } \vec{F}(s', t)}{(s' - u_0)(s' - u)} \right]
\end{aligned} \tag{A9}$$

where ω is some constant (for example 32π in [8] and $8/\pi$ in [12]) and

$$\vec{a} \stackrel{\text{def}}{=} \frac{\vec{F}(4m_\pi^2, 0)}{\omega} = \begin{pmatrix} a_0^0 \\ 0 \\ a_0^2 \end{pmatrix}. \tag{A10}$$

is a vector with, the elements of which are scattering lengths of the S_0 and S_2 wave amplitudes and are defined by the threshold expansions given in Eq. (4).

Projection of the vector $\vec{F}(s, t)$ on partial waves $f_l^I(s)$ is given by

$$\vec{f}_l(s) = \frac{1}{\omega} \int_0^1 dx P_l(x) \vec{F}(s, t), \tag{A11}$$

where

$$\vec{f}_l(s) = \begin{pmatrix} f_l^0(s) \\ f_l^1(s) \\ f_l^2(s) \end{pmatrix}, \tag{A12}$$

$P_l(x)$ are Legendre polynomials and

$$t = \frac{(s - 4m_\pi^2)(x - 1)}{2}. \tag{A13}$$

Here, due to the symmetry of the integrands, the integration limit was taken from 0 to 1 instead of from -1 to 1.

In the projection of the isospin amplitudes on partial waves it is convenient to split integrals in Eq. (A9) into low (for $s < s'_{max}$) and high energy part ($s > s'_{max}$). Value of s'_{max} is determined by limited range of applicability of phenomenological parameterizations for the input amplitudes. Typically is $s'_{max} \sim 1.4 - 2$ GeV (see for example [8] and [12]).

In the low energy part the input amplitudes on the right hand side of (A9) are expressed by

$$\vec{F}(s, t) = \omega \sum_l (2l + 1) P_l(x) \vec{f}_l(s) \tag{A14}$$

Analytical expressions for the output partial wave amplitudes are obtained from the relations (A9), (A11) and

(A14) and can be shortly written as

$$\begin{aligned}
\text{Re } \vec{f}_\ell(s) = & \xi_\ell \hat{C}_{st} \vec{a} \\
& + \sum_{l'} \int_{4m_\pi^2}^{s'_{max}} ds' \hat{K}_{\ell l'}(s, s') \text{Im } \vec{f}_{l'}(s') + \text{Re } \vec{f}_\ell^{h.e.}(s)
\end{aligned} \tag{A15}$$

where $\xi_\ell = \int_0^1 dx P_\ell(x)$ and $\text{Re } \vec{f}_\ell^{h.e.}(s)$ is the high energy part defined later. For simplicity all integrals in this equation are grouped in

$$\begin{aligned}
\hat{K}_{\ell l'}(s, s') = & (2\ell' + 1) \\
& \times \left\{ K_{\ell l'}(s, s') \hat{1} - L_{\ell l'}(s, s') \hat{C}_{su} \right. \\
& \left. + M_\ell(s, s') \hat{C}_{st} - N_\ell(s, s') \hat{C}_{st} \hat{C}_{su} \right\}.
\end{aligned} \tag{A16}$$

The $K_{ll'}(s, s')$, $L_{ll'}(s, s')$, $M_l(s, s')$ and $N_l(s, s')$ kernels read

$$K_{ll'}(s, s') = \frac{s}{\pi s'(s - s')} \int_0^1 dx P_l(x) P_{l'}(y), \tag{A17}$$

$$L_{ll'}(s, s') = \frac{s}{\pi} \int_0^1 dx P_l(x) \frac{P_{l'}(y)}{u'(u' - s)}, \tag{A18}$$

$$M_l(s, s') = \frac{1}{\pi(s' - 4m_\pi^2)} \int_0^1 dx P_l(x) \frac{t - 4m_\pi^2}{s' - t} \tag{A19}$$

and

$$N_l(s, s') = \frac{1}{\pi s'} \int_0^1 dx P_l(x) \frac{4m_\pi^2 - t}{u'} \tag{A20}$$

where $y = (u' - t)/(u' + t)$. Integrands of the kernels $L_{ll'}(s, s')$ and $N_l(s, s')$ have singularities at $u' = 0$ which for $s' < \frac{1}{2}(s + 4m_\pi^2)$ is in the range of integration. Therefore principal values are taken for these integrals.

The full analytical expressions for the $K_{ll'}^{II'}$ elements of $\hat{K}_{\ell l'}(s, s')$ are presented in Appendix B.

In the high energy parts of the integrals in Eqs. (A9), the input amplitudes can be expressed by t -channel ones $\vec{F}_t(s, t)$ using the Regge parameterizations. The relation between amplitudes in the s and t channels is

$$\vec{F}(s, t) = \hat{C}_{st} \vec{F}_t(s, t). \quad (\text{A21})$$

Following it, one can conclude that high energy partial wave amplitudes derived directly from Eqs. (A9) read

$$\begin{aligned} \text{Re } \vec{f}_\ell^{h.e.}(s) = & \int_{s'_{max}}^{\infty} ds' \left\{ \frac{s}{\pi} \int_0^1 dx P_l(x) \left[\frac{\hat{1}}{s'(s' - s)} - \frac{\hat{C}_{su}}{(s' - u_0)(s' - u)} \right] \hat{C}_{st} \text{Im} \vec{F}_t(s', t) \right. \\ & \left. + \hat{C}_{st} \left[\left(M_l(s', s) \hat{1} - N_l(s', s) \hat{C}_{su} \right) \hat{C}_{st} \text{Im} \vec{F}_t(s', 0) \right] \right\}. \end{aligned} \quad (\text{A22})$$

Finally, the full expression for given output partial wave amplitude can be written as in Eq. (1):

$$\text{Ref}_\ell^I(s) = ST_\ell^I + KT_\ell^I(s) + DT_\ell^I(s) \quad (\text{A23})$$

where for the subtracting term:

$$ST_\ell^I = \xi_l \hat{C}_{st} \vec{a} = -\frac{1}{24}(a_0^0 - \frac{5}{2}a_0^2)\delta_{I1}\delta_{l3}, \quad (\text{A24})$$

and for the kernel terms:

$$KT_\ell^I(s) = \sum_{I'=0}^2 \sum_{\ell'=0}^3 \int_{4m_\pi^2}^{s'_{max}} ds' K_{\ell\ell'}^{II'}(s, s') \text{Im } f_{\ell'}^{I'}(s'). \quad (\text{A25})$$

The driving terms $DT_\ell^I(s)$ are given by Eq. (A22).

Appendix B: Analytical expression for kernels

Following work [15] for the Roy's equations, one can express eighteen kernels $K_{\ell\ell'}^{II'}(s, s')$ in Eq. (A25) as functions of the four kernels $K_{ll'}(s, s')$ and eight combinations of the $L_{ll'}(s, s')$, $M_l(s, s')$ and $N_l(s, s')$ ones from Eqs. (A17) to (A20)

$$\begin{aligned} K_{20}^{00} &= K_{20} - I_{20}/3, & K_{20}^{02} &= -\frac{5}{3}I_{20}, \\ K_{21}^{01} &= 3I_{21}, & K_{22}^{00} &= 5(K_{22} - \frac{1}{3}I_{22}), \\ K_{22}^{02} &= -\frac{25}{3}I_{22}, & K_{23}^{01} &= 7I_{23}, \\ K_{30}^{10} &= I_{30}/3, & K_{30}^{12} &= -\frac{5}{6}I_{30}, \\ K_{31}^{11} &= 3(K_{31} - \frac{1}{2}I_{31}), & K_{32}^{10} &= \frac{5}{3}I_{32}, \\ K_{32}^{12} &= -\frac{25}{6}I_{32}, & K_{33}^{11} &= 7(K_{33} - \frac{1}{2}I_{33}), \\ K_{20}^{20} &= -I_{20}/3, & K_{20}^{22} &= K_{20} - I_{20}/6, \\ K_{21}^{21} &= -\frac{3}{2}I_{21}, & K_{22}^{20} &= -\frac{5}{3}I_{22}, \\ K_{22}^{22} &= 5(K_{22} - \frac{1}{6}I_{22}), & K_{23}^{21} &= -\frac{3}{2}I_{23}. \end{aligned}$$

where

$$\begin{aligned} I_{20} &= L_{20} - M_2 + N_2, \\ I_{21} &= L_{21} + M_2 - N_2, \\ I_{22} &= L_{22} - M_2 + N_2, \\ I_{23} &= L_{23} + M_2 - N_2, \\ I_{30} &= L_{30} + M_3 + N_3, \\ I_{31} &= L_{31} - M_3 - N_3, \\ I_{32} &= L_{32} + M_3 + N_3, \\ I_{33} &= L_{33} - M_3 - N_3. \end{aligned} \quad (\text{B1})$$

Full analytical forms of the kernels $K_{ll'}$ are

$$\begin{aligned} K_{20} &= 0, \\ K_{22} &= \frac{(s - 4m_\pi^2)s(7s - 15s' + 32m_\pi^2)}{40\pi(s - s')(s' - 4m_\pi^2)^2s'}, \\ K_{31} &= \frac{s}{8\pi s'(4m_\pi^2 - s')}, \\ K_{33} &= \frac{s}{56\pi s'(4m_\pi^2 - s')^3(s - s')} \\ &\times [-512m_\pi^6 + 8s^3 - 64m_\pi^4(s - 7s') - 35s^2s' \\ &+ 42ss'^2 - 7s'^3 + m_\pi^2(44s^2 - 56ss' - 84s'^2)]. \end{aligned} \quad (\text{B2})$$

The other (even nonzero) $K_{ll'}$ elements are not given here as they are multiplied, in (A25), by zero $f_l^I(s)$ amplitudes. It is caused by the Bose symmetry for two pions according to which the sum $I + l$ must be even. It also leads to the same condition for sum $l + l'$ in the equation (A15). The diagonal kernels $K_{22}(s, s')$ and $K_{33}(s, s')$ contain singularity at $s = s'$ which is the only type of singularities in presented equations.

In order to simplify the analytical forms of the $I_{ll'}$ elements, some terms which appear at least twice or help to reduce these formulas are given below:

$$\begin{aligned}
a_1 &= s - 4m_\pi^2, \quad a'_1 = s' - 4m_\pi^2, \quad a_2 = 2s' - 8m_\pi^2, \quad a_3 = s + 2s' - 4m_\pi^2, \quad a_4 = 2s' - s - 4m_\pi^2, \\
L_1 &= \log \left(\frac{s'}{s + s' - 4m_\pi^2} \right), \quad L_2 = \log \left(\frac{2s'}{s + 2s' - 4m_\pi^2} \right), \quad L_3 = \log \left(\frac{s + 2s' - 4m_\pi^2}{2(s + s' - 4m_\pi^2)} \right), \\
L_4 &= \log \left(\frac{2s' - 8m_\pi^2}{-s + 2s' - 4m_\pi^2} \right), \quad L_5 = \log (2(-4m_\pi^2 + s + s')),
\end{aligned} \tag{B3}$$

$$\begin{aligned}
f_1 &= 16m_\pi^4 + s^2 + 8m_\pi^2(2s - 3s') - 6ss' + 6s'^2, \\
f_2 &= 16m_\pi^4 + s^2 + 6ss' + 6s'^2 - 8m_\pi^2(s + 3s'), \\
f_3 &= 64m_\pi^6 + s^3 + 48m_\pi^4(3s - 4s') - 12s^2s' + 30ss'^2 - 20s'^3 + 12m_\pi^2(3s^2 - 12ss' + 10s'^2), \\
f_4 &= 64m_\pi^6 - s^3 - 12s^2s' - 30ss'^2 - 20s'^3 - 48m_\pi^4(s + 4s') + 12m_\pi^2(s^2 + 8ss' + 10s'^2), \\
f_5 &= 192m_\pi^6 - 3s^3 - 148s^2s' + 600ss'^2 - 480s'^3 - 16m_\pi^4(9s + 28s') + 4m_\pi^2(9s^2 - 304ss' + 360s'^2), \\
f_6 &= -64m_\pi^6 + 48(s + 4s')m_\pi^4 + s^3 + 20s'^3 + 30ss'^2 - 12(s^2 + 8s's + 10s'^2)m_\pi^2 + 12s^2s', \\
f_7 &= \frac{a_1}{a'_1} \left[9280m_\pi^6 - 16(287s + 748s')m_\pi^4 + 4(139s^2 + 896s's + 1080s'^2)m_\pi^2 + 3s^3 - 480s'^3 - 600ss'^2 - 148s^2s' \right] \\
&\quad + 48f_4L_2 + \frac{a_1f_5}{s'}.
\end{aligned} \tag{B4}$$

The analytical expressions of the $I_{ll'}$ elements can then be expressed as

$$I_{20} = -\frac{2}{3a_1^3\pi}(-3a_1a_3 - f_2L_1), \tag{B5}$$

$$\begin{aligned}
I_{21} &= \frac{3}{a_1^3a'_1\pi} \left\{ 2L_3 \left[64m_\pi^6 - 16(4s + 7s')m_\pi^4 + 4(5s^2 + 20s's + 12s'^2)m_\pi^2 - 2s^3 - 6s'^3 - 18ss'^2 - 13s^2s' \right] \right. \\
&\quad \left. + 9sa_1^2 + 24a'_1m_\pi^2a_1 - 12s^2a_1 + 72m_\pi^2sa_1 - 12a'_1s'a_1 - 18ss'a_1 - 2a'_1f_2L_2 \right\},
\end{aligned} \tag{B6}$$

$$\begin{aligned}
I_{22} &= -\frac{5}{6\pi} \left\{ \frac{3a_1(12m_\pi^2 - 3s - 4s') + 3a_1(4m_\pi^2 + 3s - 4s') + 4f_1L_4 - 4f_2L_2}{a_1^3} \right. \\
&\quad - \frac{2s}{a_1'^2} \left[\frac{9s'}{a_1} + \frac{6(32m_\pi^4 - 8(s + 4s')m_\pi^2 + s^2 + 7s'^2 + 6ss')}{a_1^2} \right] \\
&\quad + \frac{2}{a_1^3s} \left(a_1'^2 f_1L_4 + L_3 \left(256m_\pi^8 - 512(s + s')m_\pi^6 + 16(19s^2 + 52s's + 19s'^2)m_\pi^4 \right. \right. \\
&\quad \left. \left. - 8(9s^3 + 43s's^2 + 43s'^2s + 9s'^3)m_\pi^2 + 6s^4 + 6s'^4 + 42ss'^3 + 73s^2s'^2 + 42s^3s' \right) + 3 \right) \Bigg\},
\end{aligned} \tag{B7}$$

$$\begin{aligned}
I_{23} &= \frac{7}{4a_1^3\pi} \left\{ 6a_1(4m_\pi^2 + 3s - 4s') \right. \\
&\quad + \frac{a_1s}{a_1'^3} \left[5824m_\pi^6 - 48(79s + 168s')m_\pi^4 + 12(95s^2 + 348s's + 240s'^2)m_\pi^2 - 115s^3 - 312s'^3 - 792ss'^2 - 660s^2s' \right] \\
&\quad + \frac{8}{a_1'^3} L_3 \left[1024m_\pi^{10} - 256(14s + 9s')m_\pi^8 + 64(55s^2 + 108s's + 27s'^2)m_\pi^6 \right. \\
&\quad - 16(92s^3 + 333s's^2 + 252s'^2s + 37s'^3)m_\pi^4 + 4(70s^4 + 384s's^3 + 531s'^2s^2 + 236s'^3s + 24s'^4)m_\pi^2 \\
&\quad - 20s^5 - 6s'^5 - 78ss'^4 - 253s^2s'^3 - 312s^3s'^2 - 150s^4s' \Bigg] \\
&\quad \left. - 2(a_1(-36m_\pi^2 + 9s + 12s') + 4f_2L_2) \right\},
\end{aligned} \tag{B8}$$

$$\begin{aligned}
I_{30} = & \frac{1}{72 a_1^4 a_1' \pi s'} \left\{ 120 a_1' s s' a_1^2 + \left[a_1' \left(192 m_\pi^6 - 16(9s + 28s') m_\pi^4 + 4(9s^2 + 176s's + 360 s'^2) m_\pi^2 - 3s^3 \right. \right. \right. \\
& - 480s'^3 - 360ss'^2 - 148s^2 s') + s' \left(9280m_\pi^6 - 16(287s + 748s') m_\pi^4 + 4(139s^2 + 896s's + 1080s'^2) m_\pi^2 \right. \\
& \left. \left. \left. + 3s^3 - 480 s'^3 - 600ss'^2 - 148s^2 s' \right) \right] a_1 - f_6 48 a_1' s' L_1 \right\}, \tag{B9}
\end{aligned}$$

$$\begin{aligned}
I_{31} = & \frac{1}{16 a_1^4 a_1' \pi s'} \left\{ \log(s') \left(-3072 a_1' s' m_\pi^6 + 9216 a_1' s'^2 m_\pi^4 + 2304 a_1' s s' m_\pi^4 - 5760 a_1' s'^3 m_\pi^2 - 4608 a_1' s s'^2 m_\pi^2 \right. \right. \\
& - 576 a_1' s^2 s' m_\pi^2 + 960 a_1' s'^4 + 1440 a_1' s s'^3 + 576 a_1' s^2 s'^2 + 48 a_1' s^3 s') + 48 a_1' \log(2) s' f_6 \\
& + a_1 \left[s' \left(-9280 m_\pi^6 + 16(863s + 748s') m_\pi^4 + 4(-331 s^2 + 240 a_1 s - 3296 s' s - 1080 s'^2) m_\pi^2 + 93 s^3 \right. \right. \\
& + 480 s'^3 + 2520 s s'^2 + 80 a_1^2 s + 1108 s^2 s' - 120 a_1 s(s + 3s') \left. \right) \\
& + a_1' \left(-192 m_\pi^6 + 16(9s + 28 s') m_\pi^4 - 4(9s^2 - 304 s' s + 360 s'^2) m_\pi^2 + 3 s^3 + 480 s'^3 - 600 s s'^2 + 148 s^2 s' \right) \left. \right] \\
& - 48 s' \left[256 m_\pi^8 - 64(5s + 13s') m_\pi^6 + 48(3s^2 + 17 s' s + 14 s'^2) m_\pi^4 - 4(7s^3 + 63 s' s^2 + 114 s'^2 s + 50 s'^3) m_\pi^2 \right. \\
& \left. \left. + 2s^4 + 20s'^4 + 70s s'^3 + 72s^2 s'^2 + 25s^3 s' \right] L_5 - 48(4m_\pi^2 + a_1' - 2s - s') s' f_6 \log(a_3) \right\}, \tag{B10}
\end{aligned}$$

$$\begin{aligned}
I_{32} = & \frac{5}{72 a_1^4 \pi} \left\{ f_7 - \frac{48}{a_1'^2} \left[1024 m_\pi^{10} - 256(9s + 14 s') m_\pi^8 + 64(27s^2 + 108s's + 55s'^2) m_\pi^6 \right. \right. \\
& - 16(37s^3 + 252s's^2 + 333s'^2 s + 92s'^3) m_\pi^4 + 4(24s^4 + 236s's^3 + 531s'^2 s^2 + 384s'^3 s + 70 s'^4) m_\pi^2 \\
& - 6s^5 - 20s'^5 - 150ss'^4 - 312s^2 s'^3 - 253s^3 s'^2 - 78s^4 s'] L_3 + \frac{6a_1 s}{a_1'^2} \\
& \times \left[5824 m_\pi^6 - 16(73s + 668s') m_\pi^4 + 4(93s^2 + 736s's + 1220 s'^2) m_\pi^2 - 31s^3 - 640s'^3 - 820ss'^2 - 388s^2 s' \right] \left. \right\}, \tag{B11}
\end{aligned}$$

and

$$\begin{aligned}
I_{33} = & -\frac{7}{48 a_1^4 \pi} \left\{ f_7 - \frac{2a_1 s}{a_1'^3} \left[96768 m_\pi^8 - 192(277s + 1177s') m_\pi^6 + 16(1513s^2 + 8517s's + 9348 s'^2) m_\pi^4 \right. \right. \\
& - 4(1145s^3 + 10221s's^2 + 17076 s'^2 s + 9540s'^3) m_\pi^2 + 325s^4 + 3360s'^4 + 9300s s'^3 + 10368s^2 s'^2 + 4035s^3 s' \left. \right] \\
& - \frac{48}{a_1'^3} L_3 \left[4096 m_\pi^{12} - 15360(s + s') m_\pi^{10} + 1445s^3 s'^3 + 972s^4 s'^2 + 270s^5 s' + 768(23s^2 + 67s's + 23s'^2) m_\pi^8 \right. \\
& - 192(49s^3 + 257s's^2 + 257s'^2 s + 49 s'^3) m_\pi^6 + 144(18s^4 + 141s's^3 + 253s'^2 s^2 + 141s'^3 s + 18s'^4) m_\pi^4 \\
& \left. \left. - 36(10s^5 + 106 s's^4 + 283s'^2 s^3 + 283s'^3 s^2 + 106s'^4 s + 10 s'^5) m_\pi^2 + 20s^6 + 20s'^6 + 270ss'^5 + 972s^2 s'^4 \right] \right\}. \tag{B12}
\end{aligned}$$

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